Some comments on q-deformed oscillators and q-deformed $su(2)$ algebras

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Abstract

The various relations between q-deformed oscillators algebras and the q-deformed $su(2)$ algebras are discussed. In particular, we exhibit the similarity of the q-deformed $su(2)$ algebra obtained from q-oscillators via Schwinger construction and those obtained from q-Holstein-Primakoff transformation and show how the relation between $su_{\sqrt{q}}(2)$ and Hong Yan q-oscillator can be regarded as an special case of Inöuë- Wigner contraction. This latter observation and the imposition of positive norm requirement suggest that Hong Yan q-oscillator algebra is different from the usual $su_{\sqrt{q}}(2)$ algebra, contrary to current belief in the literature.

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1 Introduction

Since the Macfarlane-Biedenharn (MB) papers [1, 2] on the construction of $su_q(2)$ algebra from the q-deformed oscillator algebra à la Schwinger way, there are by now many different versions of the q-deformed algebra. However all these q-deformed oscillator algebras are not Hopf algebras except the Hong Yan type and its generalization [3, 4]. It should be stressed here that via the Schwinger construction, it is only the 'algebraic' aspect of the Hopf algebra $su_q(2)$ which can be expressed in terms of the q-oscillator algebra; the co-algebraic structure of $su_q(2)$ cannot be easily obtained from the q-oscillator algebra granted that the latter possesses a Hopf structure.

It has been claimed [5, 6] that Hong Yan (HY) Hopf algebra is the same as the $su_q(2)$ Hopf algebra and a formal relation has been established for the generators of $su_{\sqrt{q}}(2)$ and the HY oscillator algebra. Nevertheless if we impose positive norm requirement for the states, then at the representation level, the identification breaks down for some values of $|q| = 1$, since for these values, the positive norm requirement does not hold. In fact, the positive norm requirement [8] is in conflict with the truncation condition [6] imposed on the states of the oscillator so as to get finite multiplets for $su_{\sqrt{q}}(2)$. In other words, for $|q| = 1$ ($q = \mathrm{e}^{\mathrm{i}\epsilon}$, ϵ arbitrary) HY oscillator algebra is different from $su_{\sqrt{q}}(2)$ algebra. Furthermore, although $su_{\sqrt{q}}(2)$ has a $q \to 1$ limit at the coalgebra level, the coalgebraic structure for HY fails in this limit. In the following section, we summarize the q-Schwinger construction of q-deformed $su(2)$ algebra in terms of a pair of q-oscillator algebras; different q-oscillator algebras lead to different q-deformed $su(2)$ algebras. Most authors prefer to set the Casimir in their q-Schwinger construction to zero. However, one sometimes find it convenient and essential to consider *non-zero* Casimir for some physical applications[8, 9]. A natural generalization with two additonal parameters α and β is also provided. In section 3, we exhibit results for q-Holstein Primakoff (HP) transformation with *non-zero* Casimirs for the MB and HY oscillators. The results are similar to those presented in section 2. Different contractions of q-deformed $su(2)$ algebras to the various q-oscillator algebras are elucidated in section 4. In particular, we show that the relation between $su_{\sqrt{q}}(2)$ and HY q-oscillator algebras

obtained in ref [5, 6] can be regarded as a form of contraction. In the last section, we point out explicitly that at the representation level the usual $su_{\sqrt{q}}(2)$ algebra is not the same as the HY q-oscillator algebra.

We recall that the quantum universal enveloping algebra, $\mathcal{U}_q(su(2))$, was first studied by Skylanin [10] and independently by Kulish and Reshetikhin [11]. This algebra has been applied extensively to the study of the eight vertex models, the XXZ ferromagnetic and anti-ferromagnetic models and the sine-Gordon models. The universal enveloping algebra, $\mathcal{U}_q(su(2))$ is generated by three operators, $J_\pm$ and J_0 satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \tag{1a}$$

$$[J_+, J_-] = [2J_0], \tag{1b}$$

where $[x]$ denotes $\dfrac{q^x - q^{-x}}{q - q^{-1}}$.

A generalized q-deformed $su(2)$ algebra [12, 13] has also been proposed and the operators $\widehat{J}_\pm$ and $\widehat{J}_0$ satisfies a modified commutation relations

$$[\widehat{J}_0, \widehat{J}_\pm] = \pm \widehat{J}_\pm, \tag{2a}$$

$$[\widehat{J}_+, \widehat{J}_-] = \Phi(\widehat{J}_0(\widehat{J}_0 + 1)) - \Phi(\widehat{J}_0(\widehat{J}_0 - 1)), \tag{2b}$$

$$= \Psi(\widehat{J}_0) - \Psi(\widehat{J}_0 - 1), \tag{2c}$$

where the functions $\Phi(\widehat{J}_0)$ and $\Psi(\widehat{J}_o)$ are some suitably chosen functions of $\widehat{J}_0$. It has been shown in ref[13] that the imposition of hermiticity condition requires the generalized q-deformed $su(2)$ to assume the form given in eq(2).

2 q-Schwinger Construction

Traditionally, the algebra $su(2)$ can be realized in terms of a pair of bosonic creation and annihilation operators of a harmonic oscillator using the Schwinger construction. A q-analogue of this construction is given by MB[1, 2, 7]. The operators $a, a^\dagger$ and N of the q-deformed

oscillator algebra obey the relations

$$[N, a^\dagger] = a^\dagger, \qquad [N, a] = -a, \tag{3a}$$

$$aa^\dagger - q^{-1}a^\dagger a = q^N, \tag{3b}$$

$$\mathcal{C}_1 = q^N(a^\dagger a - [N]). \tag{3c}$$

This oscillator algebra does not appear to possess a Hopf structure. But a Hopf structure is possible for another version of the q-deformed oscillator which was first proposed by HY [3] in which the operators $a, a^\dagger$ and N satisfy eq(3a) and eq(3b) and

$$[a, a^\dagger] = [N+1] - [N] \tag{4a}$$

$$\mathcal{C}_2 = a^\dagger a - [N]. \tag{4b}$$

In general, these two versions of the q-deformed oscillator algebras are not equivalent[4] although the two algebras coincide on the usual 'Fock' space basis $|n>$.

Mathematically, it has always been intrinsically appealing and insightful to generalize a particular mathematical structure as much as possible[14, 15, 16]. One possible generalization of the MB algebra is to introduce two additional parameters α and β. One then defines the generalized MB (GMB) algebra[14] with the relations in eq(3b) and eq(3c) replaced by

$$aa^\dagger - q^\alpha a^\dagger a = q^{\beta N}, \tag{5a}$$

$$\mathcal{C}_3 = q^{-\alpha N}(a^\dagger a - [N]_{\alpha,\beta}), \tag{5b}$$

where $[x]_{\alpha,\beta} = \dfrac{q^{\alpha x} - q^{\beta x}}{q^\alpha - q^\beta}$ is a generalized q-bracket. A similar generalization for the HY oscillator (GHY) gives

$$[a, a^\dagger] = [N+1]_{\alpha,\beta} - [N]_{\alpha,\beta} \tag{6a}$$

$$\mathcal{C}_4 = a^\dagger a - [N]_{\alpha,\beta}. \tag{6b}$$

We next consider realization of the q-deformed $su(2)$ algebra constructed from two independent q-oscillators, $a, a^\dagger, N_a$ and $b, b^\dagger, N_b$. Following ref [1, 2, 7]

$$J_+ = a^\dagger b, \qquad J_- = b^\dagger a, \tag{7a}$$

$$J_0 = \frac{1}{2}(N_a - N_b), \qquad \mathcal{C} = \frac{1}{2}(N_a + N_b). \tag{7b}$$

Using the algebra defined in eq(3), we easily check that the operators $J_\pm$ and J_0 obey the commutation relations:

$$\begin{aligned} [J_\pm, J_0] &= \mp J_\pm & (8a) \\ [J_+, J_-] &= \{-\mathcal{C}_1(q - q^{-1}) + 1\}[2J_0] & (8b) \end{aligned}$$

Note that if we set $\mathcal{C}_1 = 0$, we obtain the result in ref [1, 2]. However, if we try to construct the realization using the algebra defined in eq(4), we arrive at the Fujikawa algebra [17] with eq(8b) replaced by:

$$\begin{aligned} [J_+, J_-] &= [2J_0] + \mathcal{C}_2\{[\mathcal{C} - J_0 + 1] \\ &\quad -[\mathcal{C} - J_0] - [\mathcal{C} + J_0 + 1] + [\mathcal{C} + J_0]\}. \end{aligned} \tag{9}$$

This is not the conventional q-deformed $su(2)$ algebra as defined in eq(1) unless $\mathcal{C}_2 = 0$, which is the case in a Fock space representation.

Analogous Schwinger construction for the GMB and GHY algebras given by eq(5) and eq(6) can be constructed. The commutation relations for the operators $\{J_+, J_-\}$ for the the GMB and GHY algebra are respectively

$$[J_+, J_-] = \{\mathcal{C}_3(q^\alpha - q^\beta) + 1\}\frac{q^{\alpha N_a + \beta N_b} - q^{\alpha N_b + \beta N_a}}{q^\alpha - q^\beta} \tag{10}$$

and

$$\begin{aligned} [J_+, J_-] &= \mathcal{C}_4\{[N_b + 1]_{\alpha,\beta} - [N_b]_{\alpha,\beta} - [N_a + 1]_{\alpha,\beta} \\ &\quad + [N_a]_{\alpha,\beta}\} + \frac{q^{\alpha N_a + \beta N_b} - q^{\alpha N_b + \beta N_a}}{q^\alpha - q^\beta}. \end{aligned} \tag{11}$$

Note that when $\beta = -\alpha$, the term $\frac{q^{\alpha N_a + \beta N_b} - q^{\alpha N_b + \beta N_a}}{q^\alpha - q^\beta}$ in eq(10) and eq(11) becomes $[2J_0]_{\alpha,\beta}$.

One can also define operators $\tilde{J}_+$, $\tilde{J}_-$ and $\tilde{J}_0$ using the relations

$$\tilde{J}_+ = q^{-(\alpha+\beta)N_b} a^\dagger b, \qquad \tilde{J}_- = b^\dagger a q^{-(\alpha+\beta)N_b}, \tag{12a}$$

$$\tilde{J}_0 = \frac{1}{2}(N_a - N_b), \qquad \tilde{\mathcal{C}} = \frac{1}{2}(N_a + N_b). \tag{12b}$$

A straightforward calculation for the GMB oscillator algebra and GHY oscillator yields

$$\tilde{J}_+\tilde{J}_- - q^{-(\alpha+\beta)}\tilde{J}_-\tilde{J}_+ = \mathcal{C}_3\{(q^\alpha - q^\beta) + 1\}[2\tilde{J}_0]_{\alpha,\beta}. \tag{13}$$

and

$$\begin{aligned} & \tilde{J}_+\tilde{J}_- - q^{-(\alpha+\beta)}\tilde{J}_-\tilde{J}_+ \\ = \; & q^{-(\alpha+\beta)N_b}\mathcal{C}_4\{[\tilde{\mathcal{C}} - \tilde{J}_0 + 1] - [\tilde{\mathcal{C}} - \tilde{J}_0] - [\tilde{\mathcal{C}} + \tilde{J}_0 + 1] \\ & +[\tilde{\mathcal{C}} + \tilde{J}_0]\} + [2\tilde{J}_0]_{\alpha,\beta} \end{aligned} \tag{14}$$

respectively. In the above Schwinger construction, the expression $[2\tilde{J}_0]_{\alpha,\beta}$ is obtained with a redefinition of the commutation relation for the operators $\tilde{J}_+$ and $\tilde{J}_-$. Note that for $\alpha+\beta = 0$, eqs(13) and (14) reduce to eqs(10) and (11) respectively.

3 q-Holstein-Primakoff Transformation

It is well-known that one can realize the undeformed $su(2)$ algebra nonlinearly with one harmonic oscillator using the HP transformation. A q-analogue of the transformation has also been studied[20]. The q-analogue of the HP transformation is defined by the relations

$$J_+ = a^\dagger\sqrt{[2j - N]}, \tag{15a}$$

$$J_- = \sqrt{[2j - N]}a, \tag{15b}$$

$$J_0 = N - j, \tag{15c}$$

where j is some c-number.

It can be checked easily that under MB q-deformed oscillators, the realization (15) leads to

$$
\begin{aligned}
[J_0, J_\pm] &= \pm J_\pm, && \text{(16a)}\\
[J_+, J_-] &= [2J_0] + \mathcal{C}_1 q^{-2J_0}; && \text{(16b)}
\end{aligned}
$$

whereas under HY oscillators, the commutation relations become

$$
\begin{aligned}
[J_0, J_\pm] &= \pm J_\pm, && \text{(17a)}\\
[J_+, J_-] &= [2J_0] + \mathcal{C}_2\{[2j - N + 1] \\
&\qquad -[2j - N]\} && \text{(17b)}\\
&= [2J_0] + \mathcal{C}_2\{[j - J_0 + 1] \\
&\qquad -[j - J_0]\}. && \text{(17c)}
\end{aligned}
$$

It is interesting to compare eq(16) and eq(17) with eq(1) and eq(9) respectively.

For the GMB and GHY oscillator algebras defined by eqs(5) and (6), one can also define the q-analogue of the HP transformations in the most obvious manner by replacing the usual q-bracket by its generalized q-bracket. It turns out that the generalized q-HP transformations are then given by the relations

$$
\begin{aligned}
\tilde{J}_+ &= q^{-\frac{\alpha+\beta}{2}N} a^\dagger \sqrt{[2j - N]_{\alpha,\beta}}, && \text{(18a)}\\
\tilde{J}_- &= \sqrt{[2j - N]_{\alpha,\beta}}\, a q^{-\frac{\alpha+\beta}{2}N}, && \text{(18b)}\\
\tilde{J}_0 &= N - j. && \text{(18c)}
\end{aligned}
$$

One easily verifies that under the GMB q-deformed oscillator, the realization turns out to be given by the relations

$$
\begin{aligned}
&[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm, && \text{(19a)}\\
&\tilde{J}_+\tilde{J}_- - q^{\alpha+\beta}\tilde{J}_-\tilde{J}_+ \\
= \; &[-2\tilde{J}_0]_{\alpha,\beta} + \mathcal{C}_3 q^{-2\tilde{J}_0\beta}; && \text{(19b)}
\end{aligned}
$$

whereas under the GHY algebra, the same computation leads to the relations

$$[\tilde{J}_0, \tilde{J}_\pm] = \pm\tilde{J}_\pm, \tag{20a}$$

$$\begin{aligned} &\tilde{J}_+\tilde{J}_- - q^{\alpha+\beta}\tilde{J}_-\tilde{J}_+ \\ = \ & q^{-(\alpha+\beta)N}\{[2j-N+1]_{\alpha,\beta} - [2j-N]_{\alpha,\beta}\}\mathcal{C}_4 \\ & \qquad\qquad +[-2\tilde{J}_0]_{\alpha,\beta}. \end{aligned} \tag{20b}$$

4 Contraction

So far we have tried to construct the q-deformed $su(2)$ from q-oscillator algebras. A somewhat reverse process, known as contraction, is possible in general. For the undeformed case, we know that the transformation[19]

$$\begin{pmatrix} h_+ \\ h_- \\ h_0 \\ 1_h \end{pmatrix} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & \frac{\eta}{2\mu^2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_0 \\ \xi \end{pmatrix} \tag{21}$$

maps the generators of $U(2)$, $J_\pm$ and J_0 with $[\mathbf{J}, \xi] = 0$ under a change of basis to the generators $h_\pm, h_0$ and 1_h such that

$$[h_0, h_\pm] = \pm h_\pm \tag{22a}$$

$$[h_+, h_-] = 2\mu^2 h_0 - \eta 1_h \tag{22b}$$

$$[\mathbf{h}, 1_h] = 0. \tag{22c}$$

One easily notes that the commutation relations eq(22) are well-defined in the limit $\mu \to 0$ despite the singularity in the transformation. For $\mu \to 0$ and $\eta \to 1$, the transformed algebra in eq(22) can be mapped isomorphically to the standard oscillator algebra. This transformation is sometimes known as the generalized Inönü-Wigner contraction.

The transformation given in eq(21) allows for a simple extension to the q-deformed case if one identifies the operators $\{h_+, h_-, h_0\}$ as the operators $\{a^\dagger, a, N'\}$, the latter satisfying

the HY algebra with $N' = N + \dfrac{1}{2}$. Further one should also demand that the operators $\{J_+, J_-, J_0\}$ obey the $q^{\frac{1}{2}}$-deformed $su(2)$ algebra. In particular, one can easily work out the commutation relations for $[h_+, h_-]$ or equivalently $[a^\dagger, a]$ explicitly to get

$$\begin{aligned}
[h_+, h_-] &= [a^\dagger, a] \\
&= \mu^2 [J_+, J_-] \\
&= \mu^2 \frac{q^{J_0} - q^{-J_0}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \\
&= \mu^2 \frac{q^{h_0} q^{-\frac{\eta}{2\mu^2}\xi} - q^{-h_0} q^{\frac{\eta}{2\mu^2}\xi}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.
\end{aligned} \tag{23}$$

However since the operators $\{h, h^\dagger, h_0\}$ or equivalently $\{a, a^\dagger, N'\}$ obey the HY algebra, one can also work out the commutation relation in eq(23) in terms of the operator h_0. An straightforward computation yields

$$\begin{aligned}
[h_+, h_-] &= [h_0 - \frac{1}{2}] - [h_0 + \frac{1}{2}] \\
&= -\frac{q^{h_0} + q^{-h_0}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}.
\end{aligned} \tag{24}$$

Consistency requirement for the expressions in eq(23) and eq(24) yields:

$$\frac{\mu^2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} q^{-\frac{\eta}{2\mu^2}\xi} = -\frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}, \tag{25a}$$

$$\frac{\mu^2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} q^{\frac{\eta}{2\mu^2}\xi} = \frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}. \tag{25b}$$

It is straightforward to solve eq(25) for μ and $\eta\xi$ giving

$$\mu = \mathrm{e}^{-\mathrm{i}\frac{\alpha'}{2}} (\frac{\mathrm{q} - 1}{\mathrm{q} + 1})^{\frac{1}{2}} \tag{26a}$$

$$\eta\xi = 2\mathrm{e}^{-\mathrm{i}\alpha'} (\frac{\mathrm{q} - 1}{\mathrm{q} + 1}) \frac{\mathrm{i}\alpha'}{\ln \mathrm{q}} \tag{26b}$$

where $\alpha' = \frac{\pi}{2} + \ell\pi$ ($\ell \in \mathbf{Z}$) and we have appropriately chosen one branch when taking the logarithm of complex number.

Thus, we observe that the relation obtained in ref [5, 6] between HY oscillator and $su_{\sqrt{q}}(2)$ algebra can be regarded as the q-analogue of the transformation given in eq(21) if we write

$$\begin{pmatrix} a_+ \\ a \\ N' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{\alpha'}{2}}(\frac{\mathrm{q}-1}{\mathrm{q}+1})^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\alpha'}{2}}(\frac{\mathrm{q}-1}{\mathrm{q}+1})^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 1 & \frac{i}{\ln q} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_0 \\ \alpha' 1 \end{pmatrix} \tag{27}$$

in which one easily identifies the quantities μ, η and ξ in eq(21) by $\mu = \mathrm{e}^{-\mathrm{i}\frac{\alpha'}{2}}(\frac{\mathrm{q}-1}{\mathrm{q}+1})^{\frac{1}{2}}$, $\eta = \frac{2i\mathrm{e}^{-\mathrm{i}\alpha'}}{\ln q}(\frac{q-1}{q+1})$ and $\xi = \alpha'$. We would like to emphasize again that the operators $J_\pm$ and J_0 in this case obey the $q^{\frac{1}{2}}$-deformed commutation relations in eq(1)[5, 6]. In the limit $q \rightarrow 1$, this transformation is again singular but again the commutation relations for the oscillator algebra are well-defined and become the undeformed oscillator algebra. Furthermore, for generic q, the coproduct, counit and antipodes for the q-deformed $su(2)$ carry directly through the transformation, endowing the HY oscillator with a Hopf structure. This Hopf structure however breaks down in the limit when $q \rightarrow 1$ whereas the Hopf structure of $su_{\sqrt{2}}(2)$ becomes cocommutative in the same limit. From refs [6, 8], it is not difficult to show that the positive norm requirement and the truncation condition for the states of the HY q-oscillator are in conflict with each other. Thus the HY q-oscillator algebra is not the same as the $su_{\sqrt{q}}(2)$ algebra.

The MB oscillator algebra can be shown via the map $a = q^{N/2}A$, $a^\dagger = A^\dagger q^{N/2}$ to be equivalent to the algebra $\mathcal{A}_q$ with operators $\{A, A^\dagger, N\}$ satisfying

$$[A, A^\dagger] = q^{-2N} \tag{28a}$$

$$[N, A] = -A \tag{28b}$$

$$[N, A^\dagger] = A^\dagger. \tag{28c}$$

In fact, Chaichian and Kulish [20] have shown that the map

$$A = \lim_{s\rightarrow\infty} \frac{(q - q^{-1})}{q^s} J_+ \tag{29a}$$

$$
\begin{aligned}
A^{\dagger} &= \lim_{s\to\infty} \frac{(q-q^{-1})}{q^s} J_- && \text{(29b)}\\
N &= s - J_0 && \text{(29c)}
\end{aligned}
$$

allows the contraction of $su_q(2)$ to the MB q-oscillator algebra. Note that this contraction clearly lifts the highest weight representation to infinity so that there exists an infinite tower of states needed for the oscillator algebra $\mathcal{A}_q$. Although this contraction does not induce a coproduct for $\{A, A^{\dagger}, N\}$, it admits a coaction $\Psi : \mathcal{A}_q \to \mathcal{A}_q \otimes SU_q(2)$ given by

$$
\begin{aligned}
\Psi(N) &= N - J_0, && \text{(30a)}\\
\Psi(A) &= Aq^{-J_0} + \sqrt{q-q^{-1}}q^{-N}J_+, && \text{(30b)}\\
\Psi(A^{\dagger}) &= A^{\dagger}q^{-J_0} + \sqrt{q-q^{-1}}q^{-N}J_-. && \text{(30c)}
\end{aligned}
$$

This coaction satisfies the associative axioms namely

$$
\begin{aligned}
(\Psi\otimes 1)\circ\Psi &= (1\otimes\Psi)\circ\Psi && \text{(31a)}\\
(1\otimes\epsilon)\circ\Psi &= 1 && \text{(31b)}
\end{aligned}
$$

where ϵ is the counit. Further, one easily checks that the homomorphism axiom is consistent, namely

$$
\Psi([x,y]) = [\Psi(x), \Psi(y)] \tag{32}
$$

where $x, y \in \{A, A^{\dagger}, N\}$. In the framework of Inöue-Wigner transformation, there seems to be a singular transformation

$$
\begin{pmatrix} A \\ A^{\dagger} \\ N \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{q-q^{-1}}}{q^s} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{q-q^{-1}}}{q^s} & 0 & 0 \\ 0 & 0 & -1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_- \\ J_+ \\ J_0 \\ 1 \end{pmatrix}. \tag{33}
$$

The contraction from $su_q(2)$ to MB oscillator algebra occurs in the singular limit $s \to \infty$, but in this case, the natural coproduct for $su_q(2)$ does not survive in this limit. This contraction

is essentially similar to the one proposed by J. Ng [21]. A different contraction proposed by Celeghini et al [5, 22] involves the transformation

$$\begin{pmatrix} B \\ B^\dagger \\ N \\ H \\ \omega \end{pmatrix} = \begin{pmatrix} \eta & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & -1 & \eta^{-2} & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \eta^{-2} \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_0 \\ K \\ \log q \end{pmatrix}. \tag{34}$$

where K is the so-called $U(1)$ generator. Under this transformation, the operators $\{B, B^\dagger, N, H\}$ obey in the limit $\eta \to 0$ the relations

$$[B, B^\dagger] = \frac{\sinh(\frac{\omega H}{2})}{\frac{\omega}{2}} \tag{35a}$$

$$[N, B] = -B, \quad [N, B^\dagger] = B^\dagger, \quad [H, N] = 0 \tag{35b}$$

$$[H, B] = \quad [H, B^\dagger] = \quad 0 \tag{35c}$$

This contraction induces a coalgebraic structure inherited from the original Hopf algebra of $su_q(2)$. The algebra generated by the operators $\{B, B^\dagger, H, N\}$ is not quite the q-deformed oscillator algebra although we can get the usual undeformed oscillator in the limit $\omega \to 0$.

5 Representations

We can gain some insights into the the linear transformation which we have encountered in the previous section by looking more closely at a representation of the HY oscillator algebra. To obtain a representation of the HY algebra[6], we note that N commutes with $a^\dagger a$ and $aa^\dagger$. As a result we can construct a vector $|\psi_0>$ which is a simultaneous eigenstate of N and $a^\dagger a$ so that

$$N|\psi_0> = \nu_0|\psi_0> \tag{36a}$$

$$a^\dagger a|\psi_0> = \lambda_0|\psi_0> \tag{36b}$$

where ν_0 and λ_0 are the corresponding eigenvalues. We shall further assume that the operator N is Hermitian so that its eigenvalue ν_0 is real.

From the eigenstate, $|\psi_0>$, one can construct other eigenstates of N by defining

$$|\psi_n> \;=\; (a^\dagger)^n|\psi_0> \tag{37a}$$

$$|\psi_{-n}> \;=\; a^n|\psi_0> \tag{37b}$$

for some positive integer n. With these definitions, one easily shows that

$$a^\dagger|\psi_n> \;=\; |\psi_{n+1}> \tag{38a}$$

$$a^\dagger|\psi_{-n}> \;=\; \lambda_{-n+1}|\psi_{-n+1}> \tag{38b}$$

$$a|\psi_n> \;=\; \lambda_n|\psi_{n-1}> \tag{38c}$$

$$a|\psi_{-n}> \;=\; |\psi_{-n-1}> \tag{38d}$$

$$N|\psi_{\pm n}> \;=\; (\nu_0 \pm n)|\psi_{\pm n}> \tag{38e}$$

where

$$\lambda_n \;=\; \lambda_0 + \frac{q^{\frac{1}{2}n} - q^{-\frac{1}{2}n}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\frac{q^{\nu_0+\frac{n}{2}} + q^{-\nu_0-\frac{n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \tag{39a}$$

$$\;=\; \lambda_0 + [n+\nu_0] - [\nu_0]. \tag{39b}$$

Note that the oscillator algebra still admits an infinite number of states and the representation at this stage is different from $\mathcal{U}_q(su(2))$ whose finite-dimensional representation requires a highest weight state. One then imposes a truncation on the tower of states and set $a|\psi_0>= 0$ giving $\lambda_0 = 0$ and $|\psi_{-n}>= 0$ for any $n > 0$. Let $|\psi_k>$ be the highest weight state so that $a^\dagger|\psi_k>= 0$ with integer $k > 0$. Since $\mathcal{C}_2 = a^\dagger a - [N] = aa^\dagger - [N+1]$, one finds by considering the action of $\mathcal{C}_2$ on $|\psi_k>$ that the following condition must be satisfied:

$$[\nu_0 + k + 1] = [\nu_0]. \tag{40}$$

For real q, $k = -1$ is the only solution, but this is not acceptable. However, for complex q with $|q| = 1$, truncation is possible. It is not difficult to solve eq(40) for ν_0 in this case.

Writing $q = \mathrm{e}^{\mathrm{i}\epsilon}$, one can show that for arbitrary ϵ, eq(40) leads to

$$\nu_0 \epsilon = \frac{-(k+1)\epsilon}{2} + (\ell + \frac{1}{2})\pi, \qquad \ell \in \mathbf{Z} \tag{41}$$

This result needs not be consistent with the condition for positivity of norms [6, 8] which by eq(39) is

$$[n + \nu_0] - [\nu_0] \geq 0 \tag{42}$$

for all integers $n \leq k$. To see this, we substitute eq(41) into the left hand side of condition (42) and see that

$$[n + \nu_0] - [\nu_0] = \frac{(-1)^\ell}{\sin \epsilon} \{\cos(\frac{k+1}{2} - n)\epsilon - \cos \frac{k+1}{2}\epsilon\}$$

which needs not be positive for arbitrary ϵ. This means that for arbitrary ϵ, we cannot proceed to identify the HY oscillator algebra with $su_{\sqrt{q}}(2)$ algebra. To identify the two algebras, we have to truncate the tower of states of the HY oscillator algebra. However, truncation and positive norm requirement can both be satisfied only for certain value of ϵ. In short, the HY oscillator algebra and $su_{\sqrt{q}}(2)$ algebra are equivalent only for certain q-values.

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